

TROPICAL MATHEMATICS

Year 2016-17

School: I.S.I.S.S. "M.Casagrande", Pieve di Soligo, Treviso - Italy

Students: Nicola Arghittu (V)
Bianca Della Libera (IV)
Gioella Lorenzon (IV)
Marina Vitali (III)
Camilla Viviani (III)
Jing Jing Xu (IV)

Teachers: Fabio Breda
Alberto Meneghello
Francesco Maria Cardano
Francesco Zampieri

Researcher: Alberto Zanardo, University of Padova - Italy

Presentation of the research topic: GÉOMÉTRIE TROPICALE

On définit deux nouvelles opérations: $a \oplus b = \min\{a; b\}$ et $a \otimes b = a + b$, par examp: $3 \oplus 7 = 3$, $3 \otimes 7 = 10$. Quelles sont les propriétés de cette nouvelle addition et multiplication? Est-ce qu'on peut "tout-faire" comme avec l'addition et la multiplication que l'on connaît? Peut-on définir la soustraction et la division?

Brief presentation of the conjectures and results obtained:

The first step is to retrace the construction of the structure $(\mathbb{C}, +, \cdot)$, starting from the Peano axioms, in order to obtain the tropical semiring $(\mathbb{R} \cup \{+\infty\}, \oplus, \odot)$.

As \mathbb{C} , such semiring is algebraically closed, that is every univariate polynomial can be factorized in linear factors, hence a tropical version of the fundamental theorem of algebra holds.

Finally, we study how tropical polynomials can give raise to tropical algebraic curves (in \mathbb{R}^2 in particular). It is difficult here to recognise the correspondence with classical geometry, that is lines, circumferences, parabolas ..., as the curve essentially depends on the number of monomials rather than the degree of the polynomial.

Contents

Introduzione	3
1 Building the structure $(\mathbb{R}, \oplus, \odot)$	4
1.1 Let me count: the basic structure $(\mathbb{N}, \oplus, \odot)$	4
1.2 Extending with quotients: the rational structure $(\mathbb{Z}, \oplus, \odot)$	9
1.3 Extending with roots: the algebraic structure $(\mathbb{Q}, \oplus, \odot)$	10
1.4 Extending with limits: the complete structure $(\mathbb{R}, \oplus, \odot)$	11
1.5 Correspondence between standard and tropical constructions	12
2 POLYNOMIALS IN $(\mathbb{R} \cup \{+\infty\}, \oplus, \odot)$	13
2.1 Monomials and polynomials	13
2.2 Tropical polynomials and tropical polynomial functions	15
2.3 Factorization of polynomials	19
2.4 The roots of a tropical polynomial	22
3 Tropical algebraic curves in \mathbb{R}^2	24
3.1 Algebraic Curves associated with a monomial	26
3.2 Algebraic Curves associated with a binomial	27
3.3 Algebraic Curves associated with a trinomial	28

Introduction

The proposed subject is the following:

On définit deux nouvelles opérations: $a \oplus b = \min\{a; b\}$ et $a \otimes b = a + b$, par exemple: $3 \oplus 7 = 3$, $3 \otimes 7 = 10$. Quelles sont les propriétés de cette nouvelle addition et multiplication? Est-ce qu'on peut "tout-faire" comme avec l'addition et la multiplication que l'on connaît? Peut-on définir la soustraction et la division?

We define two new operations: $a \oplus b = \min\{a; b\}$ and $a \otimes b = a + b$, for example: $3 \oplus 7 = 3$, $3 \otimes 7 = 10$. What are the properties of these new addition and multiplication? Can we do everything as with the usual addition and multiplication? Can we define subtraction and division?

Chapter 1

Building the structure $(\mathbb{R}, \oplus, \odot)$

Our first aim is to define a new structure, formed by the numeric set \mathbb{R} equipped with the two new operations: the tropical addition \oplus (i.e. the minimum) and the tropical multiplication \odot (i.e. the sum).

We first ask whether in the construction of such a structure it is possible to consider a numerical set which is already defined a priori - in our case the standard set \mathbb{R} of real numbers - or if such a set should be defined through the operations of the structure itself - in this case, the minimum and addition.

We face therefore with a rather challenging question: *are the sets that define the operations or vice versa?*

It is well known that the standard structure $(\mathbb{R}, +, \cdot)$ is constructed starting from the axiomatic definition of $(\mathbb{N}, +, \cdot)$ and this basic structure is then extended to $(\mathbb{Z}, +, \cdot)$, then $(\mathbb{Q}, +, \cdot)$, and finally to $(\mathbb{R}, +, \cdot)$ in order to make the numeric set *closed* by the various operations defined.

We decide then to proceed in a similar manner in the definition of the tropical structure $(\mathbb{R}, \oplus, \odot)$.

Besides other questions arise: in standard mathematics we can think of the multiplication as an iteration of the addition, but here, the tropical multiplication is not the iteration of the tropical addition. Should we perhaps consider to define the new structure also the operations which are, respectively, the iteration of the addition of tropical and tropical multiplication?

We recognize, however, that it will probably be true that the classical multiplication is the sum of classical iteration only in $(\mathbb{N}, +, \cdot)$. In fact since childhood we all know that for example $3 \cdot 5$ means adding up 3 to itself 5 times (or add 5 to itself 3 times). How can we give a meaning in the case of $\pi \cdot \sqrt{5}$, where we add π to itself for $\sqrt{5}$ times, or sum $\sqrt{5}$ to itself for π times? Probably, the "basic" definition in $(\mathbb{N}, +, \cdot)$ of the multiplication as an iteration of the addition is enlarged at the same moment in which we extend the structure over $(\mathbb{N}, +, \cdot)$.

1.1 Let me count: the basic structure $(\mathbb{N}, \oplus, \odot)$

Counting objects is the first task of a numerical set. The set \mathbb{N} exactly captures this: it gives a name for the empty collection, and a new name to every collection obtained by adding another object.

Hence we define the numerical set \mathbb{N} by using the Peano axioms:

Definition 1.1.1. *Peano Axioms*

i) *The set \mathbb{N} contains the zero:*

$$0 \in \mathbb{N}$$

ii) *The set \mathbb{N} contains the successor $s(n)$ of each of its elements n :*

$$n \in \mathbb{N} \implies s(n) \in \mathbb{N}$$

where the successor function s is defined by:

$$a) \nexists n \in \mathbb{N}. s(n) = 0$$

$$b) s(n) = s(m) \implies n = m$$

iii) Among all the sets X that enjoy properties i) and ii), \mathbb{N} is the smallest:

let X be a set such that:

- $0 \in X$
- $n \in X \implies s(n) \in X$

then $\mathbb{N} \subseteq X$.

All the sets that satisfy the Peano axioms are isomorphic, hence:

Definition 1.1.2. The set \mathbb{N} of **natural numbers** is the (only) set that satisfies the Peano axioms.

Ordering. The set \mathbb{N} is well-ordered by the relation \leq , defined by $n \leq s(n) \forall n \in \mathbb{N}$. It is a total ordering; that means the relation \leq is reflexive ($m \leq m$), anti-symmetric (if $m \leq n$ and $n \leq m$ then $n = m$), transitive (if $m \leq n$ and $n \leq k$ then $m \leq k$) and total (for every $n, m \in \mathbb{N}$ it is $n \leq m \vee m \leq n$, that is two numbers can always be compared).

Definition 1.1.3. The minimum of two numbers $m, n \in \mathbb{N}$ is defined as $\min(m, n) = \begin{cases} m \iff m \leq n \\ n \iff m > n \end{cases}$

In the usual arithmetics, \mathbb{N} is equipped with the following operations: the addition $+$, the multiplication \cdot , and the power $^{\wedge}$:

Definition 1.1.4. Standard operations on \mathbb{N}

$$\begin{array}{lll} + : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N} & \cdot : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N} & ^{\wedge} : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N} \\ \begin{cases} m + 0 = m \\ m + s(n) = s(m + n) \end{cases} & \begin{cases} m \cdot 0 = 0 \\ m \cdot s(n) = m + (m \cdot n) \end{cases} & \begin{cases} m^{\wedge 0} = 1 \\ m^{\wedge s(n)} = m \cdot (m^{\wedge n}) \end{cases} \\ m + n = \underbrace{s(\dots s(m)\dots)}_n & m \cdot n = \underbrace{m + \dots + m}_n + 0 & m^{\wedge n} = \underbrace{m \cdot \dots \cdot m}_n \cdot 1 \end{array}$$

Observe how power $^{\wedge}$ is the iteration of multiplication \cdot , which is itself the iteration of addition $+$, which is itself is the iteration of successor function s .

We introduce now the main operations of the tropical arithmetics, the tropical addition \oplus (that coincides with the minimum), and the tropical product \odot (that coincides with the standard addition).

Definition 1.1.5. Tropical operations on \mathbb{N}

$$\begin{array}{lll} \oplus : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N} & \odot : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N} & \oslash : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N} \\ m \oplus n = \begin{cases} m \iff m \leq n \\ n \iff m > n \end{cases} & \begin{cases} m \odot 0 = m \\ m \odot s(n) = s(m \odot n) \end{cases} & \begin{cases} m^{\oslash 0} = 0 \\ m^{\oslash s(n)} = m \odot (m^{\oslash n}) \end{cases} \\ = \min(m, n) & m \odot n = m + n & m^{\oslash n} = m \cdot n \end{array}$$

As before, tropical power \oslash is an iteration of tropical multiplication \odot , but now iterating tropical addition \oplus we don't obtain the tropical multiplication \otimes . We could ask what operation arises in this way; we easily see that such an operation would be quite uninteresting, as by idempency of the \oplus (minimum) for all n we get:

$$m \otimes n = \underbrace{m \oplus \dots \oplus m}_n = \min\{\dots \min\{m, m\}, \dots m\} = m$$

In the following, we are going to explore the properties enjoyed by the new operations of tropical addition (minimum) and tropical multiplication (standard addition) in the set \mathbb{N} , referring to the properties of our usual operations (standard addition and multiplication).

Definition 1.1.6. *Properties of the operations.* Let $*$, \dagger be binary operations on a set S , then:

- the operation $*$ is **commutative** on S if and only if for all $x, y \in S$ it is

$$x * y = y * x$$

- the operation $*$ is **associative** on S if and only if for all $x, y, z \in S$ it is

$$x * (y * z) = (x * y) * z$$

- the operation $*$ is **dissociative** on S if and only if for all $x, y, z, w \in S$ it is

$$x * y = x * (z * w) \quad \text{whenever} \quad y = z * w$$

- the operation $*$ is **distributive** over \dagger on S if and only if for all $x, y, z, w \in S$ we have both

$$\begin{aligned} x * (y \dagger z) &= (x * y) \dagger (x * z) && \text{(distr. on the right)} \\ (y \dagger z) * x &= (y * x) \dagger (z * x) && \text{(distr. on the left)} \end{aligned}$$

- the operation $*$ has an **identity** element $e \in S$ if and only if for all $x \in S$ we have both

$$\begin{aligned} x * e &= x && \text{(right identity)} \\ e * x &= x && \text{(left identity)} \end{aligned}$$

- the operation $*$ with identity e has **inverse** elements if and only if for every $x \in S$ there exists an element $x' \in S$ such that

$$\begin{aligned} x * x' &= e && \text{(right inverse)} \\ x' * x &= e && \text{(left inverse)} \end{aligned}$$

The usual operations $+$ and \cdot are both commutative, associative, dissociative, \cdot distributes over $+$, the identities are 0 and 1 respectively, and the inverses do not exist in \mathbb{N} (they would be $-x$ and x^{-1} respectively).

Moreover, the power $^{\wedge}$ is neither commutative, nor associative, nor dissociative, it distributes on the left over \cdot , and admits only the right identity 1; no left identity exists, as well as the inverses in \mathbb{N} (the right inverses would be the logarithms, and the left inverses the roots).

Let's explore now the properties of the tropical operations.

Theorem 1.1.7. *Tropical addition \oplus , tropical multiplication \odot , and tropical power \oslash are both commutative.*

Proof. For the tropical addition \oplus , given $m, n \in \mathbb{N}$, suppose without loss of generality¹. $m \leq n$. Then:

$$m \oplus n = \min\{m, n\} = m = \min\{n, m\} = n \oplus m$$

The case of tropical multiplication \odot follows from that of standard addition $+$, as given $m, n \in \mathbb{N}$ it is:

$$m \odot n = m + n = n + m = n \odot m$$

The same for tropical power \oslash , directly following from that of standard multiplication \cdot . □

¹The other cases are symmetric

Theorem 1.1.8. *Tropical addition \oplus , tropical multiplication \odot , and tropical power \otimes are both associative.*

Proof. For the tropical addition \oplus , given $m, n, p \in \mathbb{N}$, suppose without loss of generality $m \leq n \leq p$. Then:

$$\begin{aligned} m \oplus (n \oplus p) &= \min\{m, \min\{n, p\}\} = \min\{m, n\} = m \\ (m \oplus n) \oplus p &= \min\{\min\{m, n\}, p\} = \min\{m, p\} = m \end{aligned}$$

The case of tropical multiplication \odot follows from that of standard addition $+$, as for all $m, n, p \in \mathbb{N}$:

$$m \odot (n \odot p) = m + (n + p) = (m + n) + p = (m \odot n) \odot p$$

The same for tropical power \otimes , directly following from that of standard multiplication \cdot . □

Theorem 1.1.9. *Tropical addition \oplus , tropical multiplication \odot , and tropical power \otimes are both dissociative.*

Proof. For the tropical addition \oplus , given $m, n, p, q \in \mathbb{N}$, suppose without loss of generality $p \leq q$. Assume

$$n = p \oplus q = \min\{p, q\} = p$$

hence n and p must coincide, and thus:

$$m \oplus (p \oplus q) = \min\{m, \min\{p, q\}\} = \min\{m, p\} = m \oplus p = m \oplus n$$

The case of tropical multiplication \odot follows from that of standard addition $+$, as for all $m, n, p, q \in \mathbb{N}$, whenever $n = p \odot q = p + q$ it is:

$$m \odot n = m + n = m + (p + q) = m \odot (p \odot q)$$

The same for tropical power \otimes , directly following from that of standard multiplication \cdot . □

Theorem 1.1.10. *Tropical multiplication \odot distributes over tropical addition \oplus ; Tropical power \otimes distributes over tropical multiplication \odot .*

Proof. Given $m, n, p \in \mathbb{N}$, suppose without loss of generality $m \leq n \leq p$. Then:

$$\begin{aligned} m \odot (n \oplus p) &= m + \min\{n, p\} = m + n = m \odot n \\ (m \odot n) \oplus (m \odot p) &= \min\{m + n, m + p\} = m + n = m \odot n \end{aligned}$$

because standard addition is monotone (that is $a \leq b$ implies $a + c \leq b + c$ for every a, b, c).

The other case is symmetric, as the tropical operations are commutative.

$$(m \oplus n) \odot p = (m \odot p) \oplus (n \odot p)$$

The second statement follows directly from distributivity of \cdot over $+$ in standard arithmetics, as:

$$\begin{aligned} m \otimes (n \odot p) &= m \cdot (n + p) = m \cdot n + m \cdot p = (m \otimes n) \odot (m \otimes p) \\ (n \odot p) \otimes m &= (n + p) \cdot m = n \cdot m + p \cdot m = (n \otimes m) \odot (p \otimes m) \end{aligned}$$

□

Theorem 1.1.11. *(Freshman's dream) In \mathbb{N} , tropical power \otimes distributes over tropical addition \oplus .*

Proof. Given $m, n, p \in \mathbb{N}$, it is:

$$p^{\otimes(m \oplus n)} = \min\{m, n\} \cdot p = \min\{m \cdot p, n \cdot p\} = m^{\otimes p} \oplus n^{\otimes p}$$

because standard multiplication is monotone (that is $a \leq b$ implies $a \cdot c \leq b \cdot c$ for every a, b, c , as $c \in \mathbb{N}$ is non-negative). The other case is symmetric, as the tropical operations are commutative:

$$p^{\otimes(m \oplus n)} = p \cdot \min\{m, n\} = \min\{p \cdot m, p \cdot n\} = p^{\otimes m} \oplus p^{\otimes n}$$

□

Theorem 1.1.12. *The identity of tropical addition \oplus does not exist in \mathbb{N} .*

Proof. The identity u for tropical addition would satisfy $m = u \oplus m = \min\{u, m\}$, if and only if $m \leq u$ for all $m \in \mathbb{N}$. That is, u would be the greatest element of \mathbb{N} . Of course such an element does not exist in \mathbb{N} . □

We can add the element $u = +\infty$ to the set \mathbb{N} , provided we define the operations on it:

$$m \oplus +\infty = \min\{m, +\infty\} = m, \quad m \otimes +\infty = m + \infty = +\infty, \quad m \oslash +\infty = m \cdot \infty = +\infty$$

Note that the second expresses the tropical analogous of the usual *absorbing* property of the zero element:

$$m \cdot 0 = 0$$

Theorem 1.1.13. *The identity of tropical multiplication \odot is 0.*

Proof. It directly follows from standard addition, as $m \odot 0 = m + 0 = m$. □

Theorem 1.1.14. *The tropical addition \oplus does not admit inverses.*

Proof. The statement is trivial for \mathbb{N} , as there is no identity for tropical addition in \mathbb{N} . Even if we endow the set \mathbb{N} with the identity $+\infty$, we cannot define inverses for tropical addition.

In order to search for the inverse of $n \in \mathbb{N}$, we to find a $m \in \mathbb{N}$ such that $+\infty = m \oplus n = \min\{m, n\}$ and such a number m does not exist, because $\min\{m, n\} = +\infty$ if and only if $m = +\infty$ and $n = +\infty$. Therefore $+\infty$ is the inverse of itself, and is only element that possesses an inverse.

None of the other natural numbers admits an inverse. The problem is the loss of information caused by the tropical addition (i.e. the minimum), that makes the tropical addition \oplus not injective. For example

$$2 \oplus 2 = 2, \quad 2 \oplus 3 = 2, \quad 2 \oplus 4 = 2, \quad 2 \oplus 5 = 2, \quad \dots$$

hence it would be **impossible** to define a kind of tropical subtraction \ominus , that recovers an addend, given the sum and the other addend. □

Theorem 1.1.15. *In \mathbb{N} , the tropical addition \oplus does not admit inverses.*

Proof. Similarly to what happens with standard addition, for a number $n \in \mathbb{N}$ there is no element $m \in \mathbb{N}$ such that $0 = m \odot n = m + n$, except when $m = n = 0$. □

As for $+\infty$, we can extend the set of numbers to include the such inverse elements for tropical multiplication \otimes . Therefore, while no analogous extension from \mathbb{N} to \mathbb{Z} exists, we can find a correspondent for \mathbb{Q} .

Definition 1.1.16. (*Zero-product property*)

A structure $(S, \dagger, *)$ enjoys the zero-product property if for all $x, y \in S$ it is (here u is the identity for \dagger)

$$x * y = u \quad \implies \quad x = u \vee y = u$$

.

Theorem 1.1.17. *In \mathbb{N} , the zero-product property holds both for tropical addition and tropical multiplication.*

Proof. Consider the structure $(\mathbb{N} \cup \{+\infty\}, \oplus, \odot)$.

For the tropical product \odot (hence $u = +\infty$, the identity for \oplus) given $a, b \in \mathbb{N} \cup \{+\infty\}$ we have:

$$+\infty = m \odot n = m + n \quad \implies \quad m = +\infty \vee n = +\infty$$

because otherwise the sum of two finite numbers is always finite.

In the case of tropical sum \oplus (note that here it is $u = 0$, the identity for \odot) given $a, b \in \mathbb{N} \cup \{+\infty\}$ we have:

$$0 = m \oplus n = \min\{m, n\} \quad \implies \quad m = 0 \vee n = 0$$

Note that this last statement is a peculiarity of $\mathbb{N} \cup \{+\infty\}$, as it is the only numerical set having the minimum element 0. Therefore, it will no longer be valid in the following extended structures. \square

1.2 Extending with quotients: the rational structure $(\mathbb{Z}, \oplus, \odot)$

In the previous section we have seen that in \mathbb{N} no inverse elements exist (except for identities). What the classical arithmetics do at this point is to extend the number set in order to include them, first the inverses for addition $+$ and then for multiplication \cdot . We try to follow the same steps for the tropical context.

While there is no chance to do this for tropical addition \oplus (see Theorem 1.1.14), we can add inverses of tropical multiplication \odot , and define the set of quotients. Since tropical multiplication \odot coincides with standard addition $+$, this construction turns out to be the same of the usual set of relative numbers \mathbb{Z} .

Definition 1.2.1. *(Structure of **quotients** \mathbb{Z}) Given two natural numbers $n, m \in \mathbb{N}$, we can define their quotient as the equivalence class of couples $[(n, m)]$, considering two couples equivalent when:*

$$(n, m) \equiv (n', m') \iff n \odot m' = n' \odot m$$

We call \mathbb{Z} the set of quotients, in analogy with the usual construction.

Ordering. Likewise \mathbb{N} , also \mathbb{Z} is a total ordered set, where the order relation is defined as:

$$(\text{in } \mathbb{Z}) \quad [(n, m)] \leq [(n', m')] \quad \iff \quad n \odot m' \leq n' \odot m \quad (\text{in } \mathbb{N})$$

and hence the extension in \mathbb{Z} for the minimum is straightforward.

We can extend to \mathbb{Z} the tropical operations previously defined for \mathbb{N} .

Definition 1.2.2. Tropical operations on \mathbb{Z}

$$\begin{aligned} \oplus : \mathbb{Z} \times \mathbb{Z} &\longrightarrow \mathbb{Z} & [(m, n)] \oplus [(m', n')] &= \min([(m, n)], [(m', n')]) & (\text{in } \mathbb{Z}) \\ \odot : \mathbb{Z} \times \mathbb{Z} &\longrightarrow \mathbb{Z} & [(n, m)] \odot [(n', m')] &= [(n \odot n', m \odot m')] \\ \otimes : \mathbb{Z} \times \mathbb{Z} &\longrightarrow \mathbb{Z} & [(n, m)] \otimes [(n', m')] &= [(n^{\odot n'} \odot m^{\odot m'}, m^{\odot n'} \odot n^{\odot m'})] \end{aligned}$$

As a consequence of the redefinition of operations, it is appropriate to analyze their properties in this new structure \mathbb{Z} .

The associative, dissociative and distributive properties, still hold for the new operations in \mathbb{Z} .

Also the considerations for identity elements ($+\infty = [(+\infty, n)]$ for \oplus , $0 = [(n, n)]$ for \odot , and $1 = [(s(n), n)]$ for \otimes), for minimum operation and for tropical product are still valid in \mathbb{Z} , as well as those for the symmetrical elements with respect to a minimum.

The only difference is that in this new structure the symmetrical elements with respect to tropical multiplication \odot exist:

Definition 1.2.3. (*Symmetric element w.r.t. \odot*) Given a quotient $[(n, m)]$ in \mathbb{Z} , its symmetric (inverse) element is $[(m, n)]$ and is an element of \mathbb{Z} , and is denoted with $\odot[(n, m)]$.

The correctness of the definition is straightforward:

$$\odot[(n, m)] \odot [(n, m)] = [(m, n)] \odot [(n, m)] = [(n \odot m, n \odot m)] = [(n + m, n + m)] = [(0, 0)] = 0$$

Hence observe how the construction above embeds \mathbb{N} into the new set \mathbb{Z} : every natural number $n \in \mathbb{N}$ can be identified with the couple $\odot n = [(n, 0)]$. Then its inverse with respect to tropical multiplication \odot is $\odot n = [(0, n)]$. Furthermore this embedding is conservative on operations, for example , in the case of \odot :

$$[(n, 0)] \cdot [(n', 0)] = [(n^{\odot n'} \odot 0^{\odot 0}, 0^{\odot 0} \odot 0^{\odot 0})] = [(n \cdot n' + 0 \cdot 0, 0 \cdot 0 + 0 \cdot 0)] = [(n \cdot n', 0)]$$

in other words, the power of two natural numbers is again a natural number, having the result the same form of the two operands.

Definition 1.2.4. (*Tropical division*) Given two numbers $[(n, m)], [(n', m')] \in \mathbb{Z}$, where $[(n', m')] \neq +\infty$ we can define their division as:

$$[(n, m)] \odot [(n', m')] = [(n, m)] \odot (\odot[(n', m')]) = [n, m] \odot [(m', n')] = [(n \odot m', m \odot n')]$$

Again, the correspondence with standard arithmetics is quite elegant: while usually we are not allowed to divide by 0 (which is the identity for $+$), here tropical division is not defined on $+\infty$, i.e. the identity for \oplus . The reason is again similar, as tropical multiplication by $+\infty$ is neither surjective (its result can be solely $+\infty$, hence division by $+\infty$ could be defined only on $+\infty$ itself), nor injective (for every number q , $q \cdot +\infty = +\infty$), hence *cannot* be defined.

Being defined as a tropical product with the symmetric element, the tropical division inherits all the properties of the tropical product.

Example 1.2.1. (*Maximum*) With the availability of multiplicative inverse \odot , we can represent also the maximum operation, as the dual² of the minimum, i.e. tropical addition \oplus :

$$\begin{aligned} \max(q, p) &= -\min(-q, -p) \\ &= \odot [(\odot q) \oplus (\odot p)] \\ &= \odot [(q \oplus p) \odot (q \odot p)] \\ &= (q \odot p) \odot (q \oplus p) \\ &= p + q - \min(q, p) \end{aligned}$$

$$\max(q, p) = -\min(-q, -p) = \frac{0}{\frac{q}{\oplus} \oplus \frac{p}{\oplus}} = \frac{0}{\frac{p \oplus q}{q \odot p}} = \frac{p \odot q}{q \oplus p} = p + q - \min(q, p)$$

1.3 Extending with roots: the algebraic structure $(\mathbb{Q}, \oplus, \odot)$

In the previous section we added the multiplicative inverses, building (a tropical version of) rational numbers \mathbb{Z} . The usual construction now proceeds by considering the power operation \wedge ; its inverses are not rational numbers, in general.³ This leads to the definition of algebraic numbers, as the roots of polynomials with rational coefficients.

Again we follow the same pattern. Here the situation is fortunately simpler: the tropical power \odot , i.e. the standard multiplication \cdot , is commutative; hence its inverse (both left, tropical logarithms, and right, tropical roots \odot) coincides with standard division :

²See: de Morgan's laws.

³ Actually the situation is a bit more complex: as the operation \wedge is not commutative, we have two different inverse operations, the left inverses, i.e. logarithms, and the right ones, i.e. roots. We focus on the last ones. Allowing a root operation means to include numbers like $\sqrt[3]{3}$, but also $\sqrt[2]{3} + 1$, $\sqrt[3]{2} + \sqrt[3]{5}$, $4 \cdot \sqrt[2]{3}$ and $\sqrt[5]{\sqrt[2]{3} + 1}$; which are the roots of the polynomials $x^2 - 3$, $(x - 1)^2 - 3 = x^2 - 2x - 2$, $x^6 - 9x^4 - 10x^3 + 27x^2 - 90x - 2$, $\frac{1}{16}x^2 - 3$, and $x^{10} - 2x^5 - 2$ respectively. Hence it is equivalent to require the roots of all polynomials with rational coefficients.

Definition 1.3.1. (Structure of **roots** \mathbb{Q}) Given two rational numbers $q, p \in \mathbb{Z}$, where $p \neq 0$ we can define the p -th root of q as the equivalence class of couples $[(q, p)]$, considering two couples equivalent when:

$$(q, p) \equiv (q', p') \iff q^{\otimes p'} = q'^{\otimes p}$$

We call \mathbb{Q} the set of roots (algebraic numbers), in analogy with the usual construction.

Ordering. Likewise \mathbb{N} and \mathbb{Z} , also the set \mathbb{Q} has a total order relation \leq defined by:

$$(\text{in } \mathbb{Q}) \quad [(q, p)] \leq [(q', p')] \iff q^{\otimes p'} \leq q'^{\otimes p} \quad \text{where } p^{\otimes p'} > 0 \quad (\text{in } \mathbb{Z})$$

and hence the extension of the minimum in \mathbb{Q} is straightforward.

In addition the set \mathbb{Q} is **dense** with respect to this order relation \leq ; that is, between two distinct roots we can always find another (and thus infinite) root. Infact, given $a \neq b \in \mathbb{Q}$, suppose $a < b$, then we have:

$$a < a \otimes (b \odot a)^{\otimes c} < b \quad \text{for all } 0 < c < 1$$

We can extend to \mathbb{Q} the tropical operations previously defined for \mathbb{N} and \mathbb{Z} .

Definition 1.3.2. Tropical operations on \mathbb{Q}

$$\begin{aligned} \oplus : \mathbb{Q} \times \mathbb{Q} &\longrightarrow \mathbb{Q} & [(q, p)] \oplus [(q', p')] &= \min([(q, p)], [(q', p')]) & (\text{in } \mathbb{Q}) \\ \odot : \mathbb{Q} \times \mathbb{Q} &\longrightarrow \mathbb{Q} & [(q, p)] \odot [(q', p')] &= [(q^{\otimes p'} \odot q'^{\otimes p}, p^{\otimes p'})] \\ \oslash : \mathbb{Q} \times \mathbb{Q} &\longrightarrow \mathbb{Q} & [(q, p)] \oslash [(q', p')] &= [(q^{\otimes q'}, p^{\otimes p'})] \\ \ominus : \mathbb{Q} &\longrightarrow \mathbb{Q} & \ominus[(q', p')] &= [(\odot q, p)] \end{aligned}$$

All the properties and the considerations made for the tropical operations in the case of \mathbb{Z} still hold here.

Definition 1.3.3. (Symmetric element w.r.t. \oslash) Given a root $[(q, p)]$ in \mathbb{Z} , its symmetric (inverse) element is $[(p, q)]$. It is an element of \mathbb{Q} , and is denoted with $\oslash[(q, p)]$.

The correctness of the definition is straightforward:

$$[(q, p)]^{\oslash(\oslash[(q, p)])} = [(q, p)]^{\oslash[(p, q)]} = [(q^{\otimes p}, p^{\otimes q})] = [(1, 1)] = 1$$

Again, the construction above embeds \mathbb{Z} into the new set \mathbb{Q} : every quotient $q \in \mathbb{Z}$ can be identified with the couple $\oslash q = [(q, 1)]$. Then its inverse with respect to tropical power \oslash is $\odot q = [(1, q)]$.

Definition 1.3.4. (Tropical extraction of root) Given two numbers $[(q, p)], [(q', p')] \in \mathbb{Q}$, where $[(q', p')] \neq 0$ we can define the extraction of the $[(q', p')]$ -th root of $[(q, p)]$ as:

$$[(q, p)] \oslash [(q', p')] = [(q, p)]^{\oslash(\oslash[(q', p')])} = [(q, p)]^{\oslash[(p', q')]} = [(q^{\otimes p'}, p^{\otimes q'})]$$

With this definition, the tropical extraction of root inherits all the properties of the tropical power.

1.4 Extending with limits: the complete structure $(\mathbb{R}, \oplus, \odot)$

At this point the classical arithmetics faces the need for measuring and representing geometry. The problem is that the set of numbers, although quite large, has some holes and is not able to represent some quantities, like π , or e .

Fortunately we can approximate them arbitrarily well by using the numbers we have built so far. A Cauchy sequence is a kind of list of such approximations, and its limit is the approximated quantity.

Definition 1.4.1. A sequence $(a_n)_n$ of values from \mathbb{Q} is a Cauchy sequence if and only if⁴

$$\forall \varepsilon > 0. \exists n_\varepsilon. \forall m, n \geq n_\varepsilon. |a_m \ominus a_n| < \varepsilon$$

The limit of the sequence $(a_n)_n$ takes the value l , and we write $\lim_{n \rightarrow +\infty} a_n = l$ if and only if

$$\forall \varepsilon > 0. \exists N_\varepsilon. \forall n \geq N_\varepsilon. |a_n \ominus l| < \varepsilon$$

A numeric set is said **complete** if it contains the limits of all of its Cauchy sequences.

Definition 1.4.2. (Structure of **limits** \mathbb{R}) The set \mathbb{R} of **real numbers** is the set of the limits l of all the Cauchy sequences that can be built in \mathbb{Q} .

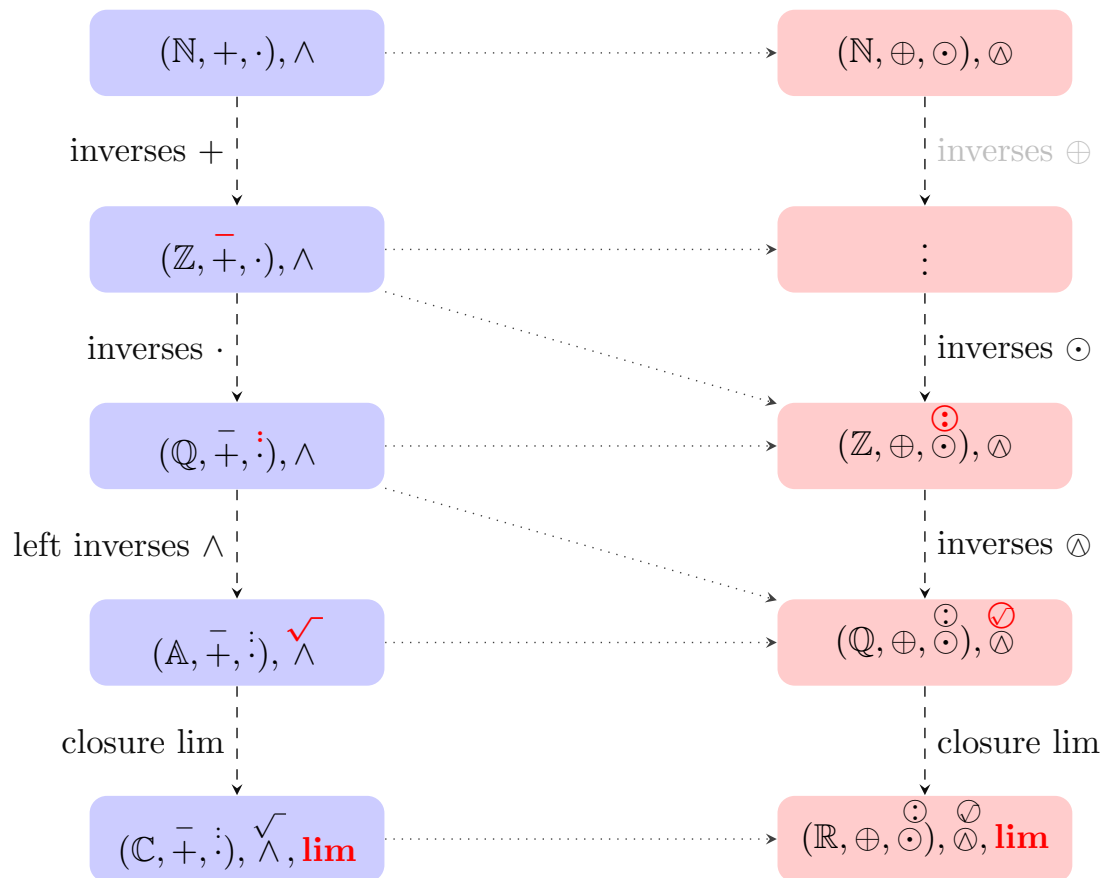
We can add the tropical additive identity $+\infty$ and obtain the tropical semiring $(\mathbb{R} \cup \{+\infty\}, \oplus, \odot)$. The correspondent usual construction would lead to the complete, algebraically closed set of complex numbers \mathbb{C} . In the following chapter we'll see how \mathbb{C} actually shares these properties with the tropical semiring (e.g in polynomial factorization).

1.5 Correspondence between standard and tropical constructions

We constructed the tropical structure $(\mathbb{R}, \oplus, \odot)$ through subsequent extensions, starting from $(\mathbb{N}, \oplus, \odot)$, similarly to the standard construction that leads to the structure $(\mathbb{C}, +, \cdot)$. We summarize correspondences and discrepancies in this table:

Standard Arithmetics

Tropical Arithmetics



⁴ The absolute value $|\cdot|$ is defined as: $|a| = \begin{cases} a & \text{if } a \geq 0 \\ \ominus a & \text{if } a < 0 \end{cases}$

Chapter 2

POLYNOMIALS IN $(\mathbb{R} \cup \{+\infty\}, \oplus, \odot)$

In this second part we will deal with monomials and polynomials in $(\mathbb{R} \cup \{+\infty\}, \oplus, \odot)$ and then with their factorization.

Prime matter was how to define monomials and polynomials, that is to follow our classical mathematics or not. The classical mathematics defines the polynomial as a sum of products and not a product of sums.

Given the validity of the multiplication's distributive property over addition and the fundamental theorem of algebra, according to the classical mathematics these two expressions correspond.

So we thought about operating the same way in tropical geometry, for we already proved the validity of the distributive property in the former chapter and for we will be able to prove the fundamental theorem of algebra.

We also choose to use the same precedences of the classical mathematics (multiplication over addition).

Example:

$$a \odot x^{\otimes n} \oplus b \odot x^{\otimes m} \oplus c = (a \odot x^{\otimes n}) \oplus (b \odot x^{\otimes m}) \oplus c$$

We considered worthwhile lingering on the analysis of the equality between two polynomials.

In classical mathematics the principle of identity of polynomials states that two polynomials, reduced in normal form, are equal if and only if they have the same degree and the coefficients of the same-degree-terms are equal.

We will analyse if such principle is valid also for tropical geometry.

2.1 Monomials and polynomials

Definition 2.1.1. *We are given $x_1 \dots x_n$ as variables and they are represented in the tropical semiring $(\mathbb{R} \cup \{+\infty\}, \oplus, \odot)$. A tropical monomial is the tropical product between variables, where repetitions are allowed and there can be tropical powers.*

By using the commutative and associative property we can order the products and write the monomial in normal form.

Example:

$$3 \odot x \odot y \odot x \odot x \odot y = 3 \odot x^{\otimes 3} \odot y^{\otimes 2}$$

Definition 2.1.2. *A tropical polynomial is the tropical sum of tropical monomials.*

Example:

$$2 \odot x^{\otimes 4} \odot y^{\otimes 2} \oplus 4 \odot x \odot y^{\otimes 6}$$

Operations. Now we are going to study the operations between tropical monomials and polynomials lingering on those with only one variable.

- Tropical sum between two monomials:

1. similar monomials:

Example:

$$3 \odot x^{\otimes 2} \oplus 5 \odot x^{\otimes 2} = 3 \odot x^{\otimes 2}$$

Generalisation:

$$a \odot x^{\otimes e} \oplus b \odot x^{\otimes e} = (a \oplus b) \odot x^{\otimes e} \Rightarrow \text{it is an internal operation.}$$

2. with different degree:

Esempio:

$$4 \odot x^{\otimes 7} \oplus 6 \odot x^{\otimes 5} \Rightarrow \text{Polynomial made up of two addends}$$

Generalisation:

$$a \odot x^{\otimes e} \oplus b \odot x^{\otimes e'} \Rightarrow \text{Polynomial made up of two addends}$$

• Tropical sum between two polynomials:

1. If the monomials of the first polynomial have the same degree of at least one of the monomials of the following polynomials:

Example 1:

$$(2 \odot x^{\otimes 4} \oplus 10 \odot x^{\otimes 2}) \oplus (5 \odot x^{\otimes 4} \oplus 6 \odot x^{\otimes 2}) = 2 \odot x^{\otimes 4} \oplus 6 \odot x^{\otimes 2} \Rightarrow \text{Polynomial made up of two monomials}$$

Generalisation:

$$(a \odot x^{\otimes e} \oplus b \odot x^{\otimes e'}) \oplus (c \odot x^{\otimes e} \oplus d \odot x^{\otimes e'}) = (a \oplus c) \odot x^{\otimes e} \oplus (b \oplus d) \odot x^{\otimes e'} \Rightarrow \text{polynomial} \Rightarrow \text{it's an internal operation "made up of a number of monomials equal the ones that form the polynomials of departure"}$$

Example 2:

$$(2 \odot x^{\otimes 6} \oplus 10 \odot x^{\otimes 3}) \oplus (5 \odot x^{\otimes 6} \oplus 6 \odot x^{\otimes 2}) = 2 \odot x^{\otimes 6} \oplus 10 \odot x^{\otimes 3} \oplus 6 \odot x^{\otimes 2} \Rightarrow \text{Polynomial made up of three monomials}$$

Generalisation:

$$(a \odot x^{\otimes e} \oplus b \odot x^{\otimes e'}) \oplus (c \odot x^{\otimes e} \oplus d \odot x^{\otimes e''}) = (a \oplus c) \odot x^{\otimes e} \oplus b \odot x^{\otimes e'} \oplus d \odot x^{\otimes e''} \Rightarrow \text{polynomial} \Rightarrow \text{it is an internal operation.}$$

2. if all the monomials have a different degree:

Example:

$$(3 \odot x^{\otimes 6} \oplus 2 \odot x^{\otimes 2}) \oplus (6 \odot x^{\otimes 5} \oplus 8 \odot x) \Rightarrow \text{Polynomial "made up of four monomials"}$$

Generalisation:

$$a \odot x^{\otimes e} \oplus b \odot x^{\otimes e'} \oplus c \odot x^{\otimes e''} \oplus d \odot x^{\otimes e'''} \Rightarrow \text{polynomial "made up of a number of monomials equal to the sum of the monomials of departure"}$$

• Tropical product between two monomials:

Example:

$$4 \odot x^{\otimes 7} \odot 6 \odot x^{\otimes 5} = 10 \odot x^{\otimes 12}$$

Generalisation:

$$a \odot x^{\otimes e} \odot b \odot x^{\otimes e'} = (a \odot b) \odot x^{\otimes e \odot e'}$$

We can notice that the tropical product among m monomials, has always as result only one monomial which degree is equal to the tropical product of the factors' degrees.

• Tropical product between two polynomials:

Example:

$$(3 \odot x^{\otimes 6} \oplus 2 \odot x^{\otimes 2}) \odot (6 \odot x^{\otimes 5} \oplus 8 \odot x^{\otimes 1}) = 9 \odot x^{\otimes 11} \oplus 11 \odot x^{\otimes 7} \oplus 8 \odot x^{\otimes 7} \oplus 10 \odot x^{\otimes 3} = 9 \odot x^{\otimes 11} \oplus 8 \odot x^{\otimes 7} \oplus 10 \odot x^{\otimes 3}$$

In tropical geometry, as we already pointed out in the previous chapter, the product's distributive property holds upon the sum.

We point out some expressions that will be particularly frequent in the following part in order to avoid misunderstanding:

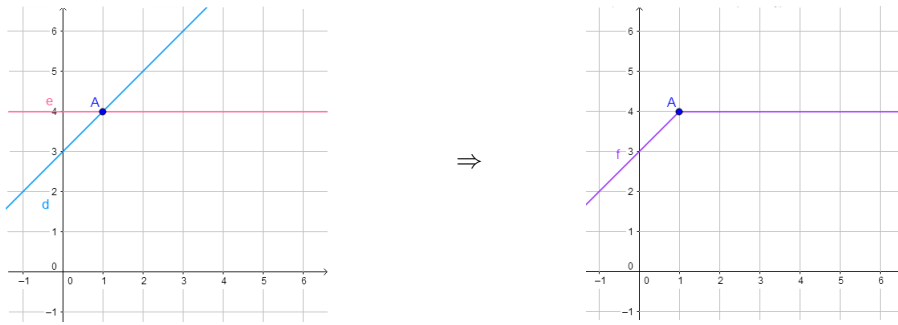
$$0 \odot x = x \quad 1 \odot x = 1 + x \quad x^{\otimes 0} = 0 \quad x^{\otimes 1} = x$$

2.2 Tropical polynomials and tropical polynomial functions

Considering the first degree tropical polynomial $3 \odot x \oplus 4$, in classical mathematics there is a function associated to it, which equation is

$$f(x) = \min\{x + 3, 4\}$$

In the cartesian coordinate system this function can be represented by dividing it in two parts $y = x + 3$ and $y = 4$, in order to study them individually. Then, considering that in the polynomial of departure $y = x + 3$ and $y = 4$ are linked from the operation of minimum, we will only consider the tracts characterized by the smallest y .



The point A is the intersection between the two straight lines and it is the point from where we begin to consider the straight line with null gradient rather than 1. To find it we solve the system:

$$\begin{cases} y = x + 3 \\ y = 4 \end{cases}$$

whose solution is the point $A(1; 4)$.

Observing the graph it is comprehensible that what the representation of the function associated to the first degree tropical polynomial shows is a piecewise function that in classical mathematics can be written this way:

$$y = \begin{cases} x + 3 & \text{se } x < 1 \\ 4 & \text{se } x \geq 1 \end{cases}$$

In general, given the polynomial $a \odot x \oplus b$ with $a, b \in \mathbb{R}$ the two straight lines obtainable by dividing the function associated to the tropical polynomial are:

- $y = x + a$, translating it into standard mathematics we can see that it is the I and III quadrant's bisector shifted of a vector $\vec{v}(0, a)$.
- $y = b$ which is a straight line with null slope.

To find the point of intersection between the two straight lines, it is needed to form a system with the two equations:

$$\begin{cases} y = x + a \\ y = b \end{cases} \quad / \quad \begin{cases} x + a = b \\ y = b \end{cases} \quad \Rightarrow \quad \begin{cases} x = b - a \\ y = b \end{cases}$$

and the piecewise function obtained is this:

$$y = \begin{cases} a + x & \text{se } x < b - a \\ b & \text{se } x \geq b - a \end{cases}$$

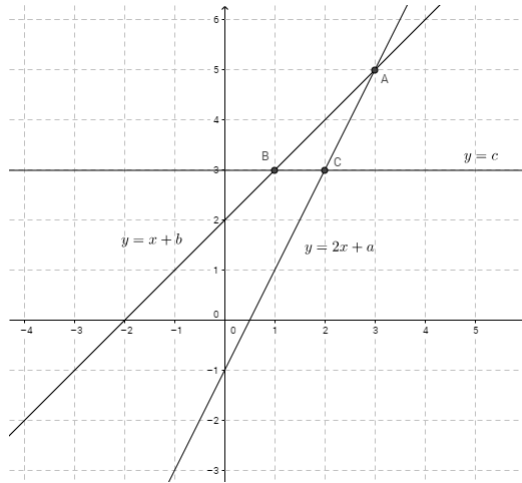
Let's now consider a second degree tropical polynomial:

$$a \odot x^{\otimes 2} \oplus b \odot x \oplus c$$

The equation of the function of classical mathematics associated to this tropical polynomial is:

$$y = \min\{a + 2x, b + x, c\}$$

To represent such function we have to plot independently the 3 functions in a Cartesian coordinate system: $y = 2x + a$, $y = x + b$ and $y = c$.



We call A the intersection between the first and the second straight line:

$$\begin{cases} y = 2x + a \\ y = x + b \end{cases} \quad / \quad \begin{cases} 2x + a = x + b \\ x = b - a \end{cases} \quad \Rightarrow \quad A(b - a; 2b - a)$$

We call B the intersection between the second straight line, whose gradient is 1, and the third straight line, whose slope is null:

$$\begin{cases} y = x + b \\ y = c \end{cases} \quad / \quad \begin{cases} x + b = c \\ x = c - b \end{cases} \quad \Rightarrow \quad B(c - b; c)$$

We call C the intersection between the first and the last straight line:

$$\begin{cases} y = 2x + a \\ y = c \end{cases} \quad / \quad \begin{cases} 2x + a = c \\ x = \frac{c-a}{2} \end{cases} \quad \Rightarrow \quad C\left(\frac{c-a}{2}; c\right)$$

As the straight lines are three and they always have the same slope, we can obtain three different graphs according to the position of the intersections:

1. if $x_A > x_B \Rightarrow b - a > c - b \Rightarrow 2b > c + a$



Fig 1.

2. if $x_A = x_B \Rightarrow b - a = c - b \Rightarrow 2b = c + a$



Fig 2.

3. if $x_A < x_B \Rightarrow b - a < c - b \Rightarrow 2b < c + a$



Fig 3.

So every tropical polynomial in one variable represents a function $\mathbb{R}^n \rightarrow \mathbb{R}$. While evaluating this function in classic mathematics, we obtain the minimum of a collection of linear functions.

$$p(x) = \min\{q_0, q_1 + x, q_2 + 2x, \dots, q_n + nx\}$$

This functions satisfy the following properties:

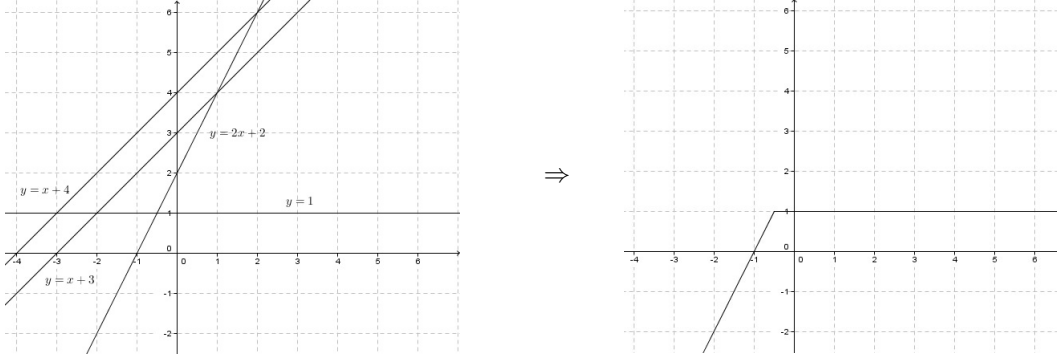
- p is continuous,
- p is a piecewise function and the number of $n + 1$ pieces is finite,
- p is concave, that is $p\left(\frac{x+y}{2}\right) \geq \frac{p(x)+p(y)}{2}$ for all $x, y \in \mathbb{R}^n$.

So every tropical polynomial in one variable is associated to a single, piecewise-defined, continuous and concave function of classic mathematics with integer-gradient-pieces that we will name **tropical polynomial function**.

The function that associates a tropical polynomial to the tropical polynomial function is surjective but not injective.

Indeed, for example, the polynomial $2 \odot x^{\otimes 2} \oplus 3 \odot x \oplus 1$ is associated to the function $f(x) = \min\{2 + 2x, 3 + x, 1\}$ while the polynomial $2 \odot x^{\otimes 2} \oplus 3 \odot x \oplus 1$ is associated to the function $g(x) = \min\{2 + 2x, 4 + x, 1\}$.

So it is visible that the two polynomials are different but their associated functions are the same, as the graphs show.



Let's remember the following definitions of classical mathematics:

Definition 2.2.1. Two real functions with real variables f and g are equal if and only if $f(x) = g(x)$ for each $x \in \mathbb{R}$.

Definition 2.2.2. Two polynomials with the same degree p and q are equal if and only if all the coefficients of the same-degree monomials are equal.

This leads us to the idea of creating some **classes of equivalence** made up with polynomials that are associated to the same tropical polynomial function.

In order to specify this idea we will state and prove it for the following second-degree tropical polynomials (the idea does not apply to those of first degree, in fact the function p is bijective, while it does apply to second or higher-degree polynomials):

Theorem 2.2.3. A second-degree tropical polynomial is an expression $a \odot x^{\otimes 2} \oplus b \odot x \oplus c$ con $a, b, c \in \mathbb{R}$, such that

$$a \odot x^{\otimes 2} \oplus b \odot x \oplus c = a' \odot x^{\otimes 2} \oplus b' \odot x \oplus c' \iff \begin{cases} a = a' \quad \wedge \quad b = b' \quad \wedge \quad c = c' \quad \wedge \quad b^{\otimes 2} < a \odot c \\ \text{or} \\ a = a' \quad \wedge \quad c = c' \quad \wedge \quad b^{\otimes 2} \geq a \odot c \quad \wedge \quad b'^{\otimes 2} \geq a \odot c \end{cases}$$

Proof:

Case 1: $b^{\otimes 2} > a \odot c$

The straight line with unitary slope encounters $y = c$ before the straight line $y = 2x + a$, so $y = x + b$ does not affect the final graph (the one in which we consider only the edges with lower y) hence, the polynomial of departure could be matched to another polynomial made up by a different monomial of first degree if the polynomial satisfies the condition that $b^{\otimes 2} > a \odot c$ (see Figure 1).

This means that different polynomials with a and c which respect the inequality $b^{\otimes 2} > a \odot c$ generate equal functions in the Cartesian coordinate system.

Case 2: $b^{\otimes 2} = a \odot c$

In this case all the three lines are intersecting in the same point so even in this case we can avoid considering the straight line with intermediate slope. This means that a polynomial in which $b^{\otimes 2} = a \odot c$

and any other polynomial with the same a and c and that satisfies the condition $b^{\otimes 2} = a \odot c$ are equal (see Figure 2)

Case 3: $b^{\otimes 2} < a \odot c$

In this case all the straight lines influence the graph so, the functions that derive from different polynomials which respect the inequality $b^{\otimes 2} < a \odot c$ are different (see Figure 3). □

Distinct polynomials can represent the same function. For example, when we have $b^{\otimes 2} \geq a \odot c$ we can claim that the polynomial $a \odot x^{\otimes 2} \oplus b \odot x \oplus c$ is equivalent to the polynomial $a \odot x^{\otimes 2} \oplus c$, i.d. they are associated to the same function. So we could name the latter as **representative of the equivalence class**:

$$[a \odot x^{\otimes 2} \oplus c] = \{a \odot x^{\otimes 2} \oplus b \odot x \oplus c \mid b^{\otimes 2} \geq a \odot c\}$$

Remember that these polynomials are not equivalent if $b^{\otimes 2} < a \odot c$ then

$$[a \odot x^{\otimes 2} \oplus b \odot x \oplus c] = \{a \odot x^{\otimes 2} \oplus b \odot x \oplus c\}$$

i.d. the only polynomial that is equivalent to the polynomial is the polynomial itself, which is the representative of the equivalence class.

So we can claim that the function that associates the tropical polynomials' classes of equivalence to the tropical polynomial functions is bijective.

Non-injectivity can be found in classical mathematics too if we consider a set made up by a finite number of elements. In fact recalling the following result

Theorem 2.2.4 (Fermat's little theorem). *If p is a prime number, then for any integer a*

$$a^p \equiv a \pmod{p}$$

For example, if we operate in the set \mathbb{Z}_3 , made up by 0, 1, and 2, and the monomial x , we could associate to the latter the function $f(x) = x$ and try to substitute the three elements in order to associate 0 to 0, 1 to 1 and 2 to 2.

It's the same if we operate similarly with the polynomial x^3 . Even in this case we obtain a function $g(x) = x^3$ which results equal to f because $g(0) = 0$, $g(1) = 1$ and $g(2) = 8 \equiv 2 \pmod{3}$.

In other words, polynomials x and x^3 are different but their functions, $f(x) = x$ and $g(x) = x^3$, appear equal because for every $x \in \mathbb{Z}_3$ $f(x) = g(x)$. Therefore we can state that a biunivocal relation between function and polynomial does not exist, while it does in tropical geometry.

2.3 Factorization of polynomials

Another important issue, in comparison with classical mathematics, is the factorization of polynomials. With very similar reasoning we can come to very interesting results.

First of all we can review the technique of factoring polynomials by taking a common factor which comes from the distributive property of multiplication over addition.

Theorem 2.3.1 (Factoring out the greatest common factor). *Given a polynomial $a \odot x \oplus b$ with $a, b \in \mathbb{R}$, the following equality is valid*

$$a \odot x \oplus b = a \odot (x \oplus (b \odot a))$$

Proof. Applying the tropical distributive property of multiplication over addition $a \odot (b \oplus c) = (a \odot b) \oplus (a \odot c)$ we obtain

$$a \odot (x \oplus (b \odot a)) = a \odot x \oplus a \odot (b \odot a) = a \odot x \oplus b$$

□

There is also the possibility of factoring a polynomial by grouping:

Theorem 2.3.2 (Factoring by grouping). *Given a polynomial $a \odot x \oplus b \odot x \oplus a \odot y \oplus b \odot y$ with $a, b \in \mathbb{R}$, the following equality is valid*

$$a \odot x \oplus b \odot x \oplus a \odot y \oplus b \odot y = (a \oplus b) \odot (x \oplus y)$$

For the factorization of the second-degree polynomials there is theorem similar to the one valid for the standard algebra:

Theorem 2.3.3. *Given a polynomial of second degree $a \odot x^{\otimes 2} \oplus b \odot x \oplus c$ with a, b, c real numbers, it can be factorized in the form*

$$a \odot (x \oplus (b \odot a)) \odot (x \oplus (c \odot b)) \quad \text{if} \quad b^{\otimes 2} < c \odot a$$

or in the form

$$a \odot (x \oplus (c \odot a) \oslash 2)^{\otimes 2} \quad \text{if} \quad b^{\otimes 2} \geq c \odot a$$

Proof. If $b^{\otimes 2} < c \odot a$, to the tropical polynomial $a \odot x^{\otimes 2} \oplus b \odot x \oplus c$ we can associate the function

$$\min\{2x + a, x + b, c\}$$

According to what we saw before we can factor out a

$$a + \min\{2x, x + b - a, c - a\}$$

Remember that the case we are considering the case whereby $2b < c + a$, so $b - a < c - b$ and so adding x to both sides we obtain $x + b - a < x + c - b$.

Now we have to minimise $x + b - a$, and if it is for sure minor than $x + c - b$ we can add the the latter to the terms that ha to be minimised because it surely does not have any influence:

$$a + \min\{2x, x + b - a, c - a\} = a + \min\{2x, x + c - b, x + b - a, c - a\}$$

Applying the factoring by grouping we obtain

$$a + \min\{x, b - a\} + \min\{x, c - b\}$$

that rewritten in tropical geometry becomes:

$$a \odot (x \oplus (b \odot a)) \odot (x \oplus (c \odot b))$$

If $b^{\otimes 2} = c \odot a$ then $b - a = c - b$ the three straight lines intersect in the same point, thus A, B, C coincide. Hence $b - a = c - b = \frac{c - a}{2}$ so

$$a \odot (x \oplus (b \odot a)) \odot (x \oplus (c \odot b)) = a \odot (x \oplus (c \odot a) \oslash 2) \oslash 2$$

If $b^{\otimes 2} > c \odot a$, Finally, as previously shown we can affirm that

$$a \odot x^{\otimes 2} \oplus b \odot x \oplus c = a \odot x^{\otimes 2} \oplus c$$

and for the Freshman's Dream

$$a \odot x^{\otimes 2} \oplus b \odot x \oplus c = a \odot x^{\otimes 2} \oplus c = a \odot (x \oplus (c \odot a) \oslash 2)^{\otimes 2}$$

□

Similarly we can factorize also higher-degree polynomials.

We can state that every tropical polynomial in one variable can be written as a tropical product between first-degree tropical polynomials. Hence **Fundamental theorem of algebra** is valid also in the tropical version.

Let's consider a general n -degree polynomial:

$$a_n \odot x^{\otimes n} \oplus a_{n-1} \odot x^{\otimes n-1} \oplus a_{n-2} \odot x^{\otimes n-2} \oplus \dots \oplus a_1 \odot x \oplus a_0$$

If we cannot simplify any of the monomials we can easily find its nodes and consequently we can find a way to factorize the polynomial.

Every node is given by the intersection of straight lines drawn to represent the polynomial function associated to the tropical polynomial.

The tropical monomials functions to whom are associated the functions that we will discuss have consecutive degrees. Our aim is to find the intersection between the straight lines with consecutive slope. For example between straight lines with unitary slope and slope 2 and so on.

So we must plan a system of classical mathematics' functions, associated to the tropical monomials:

$$\begin{cases} y = n \cdot x + a_n \\ y = (n-1) \cdot x + a_{n-1} \end{cases} \quad \begin{cases} // \\ n \cdot x + a_n = n \cdot x - x + a_{n-1} \end{cases} \quad \begin{cases} // \\ x = a_{n-1} - a_n \end{cases}$$

The x -coordinate of the first intersection is the difference between the coefficients of the second and first monomial.

In wider terms, we can associate to whatever pair of monomials two classical algebraic functions. The first function's degree is k , and the second one's is minor than 1. We write these as: $y = (n-k) \cdot x + a_{n-k}$ e $y = (n-k-1) \cdot x + a_{n-k-1}$.

The intersection between the straight lines is given by the following system:

$$\begin{cases} y = (n-k) \cdot x + a_{n-k} \\ y = (n-k-1) \cdot x + a_{n-k-1} \end{cases} \quad \begin{cases} // \\ n \cdot x - k \cdot x + a_{n-k} = n \cdot x - k \cdot x - x + a_{n-k-1} \end{cases}$$

$$\begin{cases} // \\ x = a_{n-k-1} - a_{n-k} \end{cases}$$

The system shows that all the nodes have as x -coordinate the difference between tropical monomials' coefficients, whence the functions origin.

Generalizing the factorization:

Theorem 2.3.4. *Given a tropical n -degree polynomial, which cannot be simplified:*

$$a_n \odot x^{\otimes n} \oplus a_{n-1} \odot x^{\otimes n-1} \oplus a_{n-2} \odot x^{\otimes n-2} \oplus \dots \oplus a_1 \odot x \oplus a_0$$

it can be factorized as:

$$a_n \odot (x \oplus (a_{n-1} \odot a_n)) \odot \dots \odot (x \oplus (a_{n-k-1} \odot a_{n-k})) \odot \dots \odot (x \oplus (a_0 \odot a_1))$$

□

We can say that the **Fundamental theorem of algebra** holds also in the tropical version.

Theorem 2.3.5. *The tropical semiring is algebraically closed.*

OBSERVATION: Similarly to our mathematics we can deduct that defining the monomials as products and the polynomials as sums or vice versa is the same since through the factorization and the distributive property we can switch from one form to the other.

Warning. The factorization of tropical polynomials with more variables in irreducible tropical polynomials is not unique, for example:

$$(0 \odot x \oplus 0) \odot (0 \odot y \oplus 0) \odot (0 \odot x \odot y \oplus 0) = (0 \odot x \odot y \oplus 0 \odot x \oplus 0) \odot (0 \odot x \odot y \oplus 0 \odot y \oplus 0)$$

What we displayed until now gives us the idea of defining the zeros of a tropical polynomial, similarly to the way there are defined in the classical algebra.

2.4 The roots of a tropical polynomial

In classical mathematics there is a theorem that is similar to the one proved above:

Theorem 2.4.1. *Given a second-degree trinomial $ax^2 + bx + c$ with real numbers a, b, c , it can be factorized, in the set of complex numbers, in the form:*

$$ax^2 + bx + c = a(x - x_1)(x - x_2)$$

where x_1 and x_2 are the solutions of the quadratic equation $ax^2 + bx + c = 0$. The solutions are equal if $b^2 - 4ac = 0$, otherwise they are distinct.

The values x_1 and x_2 are called roots of the polynomial.

Definition 2.4.2. *If $p(x)$ is a polynomial with coefficients in A , we say that an element c of A is a **root of the polynomial** if $p(c) = 0$.*

So in classical mathematics the roots of the polynomial are all the numbers c of A for whom $p(c) = 0$ and the ones for the factorization of the polynomial.

Here comes the idea to call roots of the tropical polynomial the useful values to factorize the tropical polynomial itself.

According to the theorem 2.3.3 we call roots of the second-degree polynomial the values $b \odot a$ and $c \odot b$, that are the x -coordinates of the piecewise function's nodes.

If we substitute these values to the x variable of the polynomial we don't obtain 0 (the zero for the tropical addition corresponds to the infinite). Let's try to substitute these values to the variable $x_1 = b \odot a$ or $x_2 = c \odot b$. What we obtain is:

$$\begin{cases} x = b \odot a \\ a \odot x^{\odot 2} \oplus b \odot x \oplus c \end{cases} \quad \begin{cases} x = b \odot a \\ a \odot (b \odot a)^{\odot 2} \oplus b \odot (b \odot a) \oplus c \end{cases} \quad \begin{cases} x = b \odot a \\ a \odot b^{\odot 2} \odot a^{\odot 2} \oplus b \odot b \odot a \oplus c \end{cases}$$

Hence

$$\begin{cases} x = b \odot a \\ b^{\odot 2} \odot a \oplus b^{\odot 2} \odot a \oplus c \end{cases}$$

Thus if we substitute the x -coordinates of the first node to the polynomial we obtain two equal addends ($b^{\otimes 2} \odot a$). Moreover, since $b^{\otimes 2} < c \odot a \rightarrow c > b^{\otimes 2} \odot c$ these two terms will surely be less than the third.

If we substitute the root zero $c \odot b$:

$$\begin{cases} x = c \odot b \\ a \odot x^{\otimes 2} \oplus b \odot x \oplus c \end{cases} \quad \begin{cases} x = c \odot b \\ a \odot (c \odot b)^{\otimes 2} \oplus b \odot (c \odot b) \oplus c \end{cases} \quad \begin{cases} x = c \odot b \\ a \odot c^{\otimes 2} \odot b^{\otimes 2} \oplus b \odot c \odot b \oplus c \end{cases}$$

Hence

$$\begin{cases} x = c \odot b \\ a \odot c^{\otimes 2} \odot b^{\otimes 2} \oplus c \oplus c \end{cases}$$

Also in this case we obtain two equal terms (c) and we know that they surely are less than the first: $b^{\otimes 2} < c \odot a \rightarrow b^{\otimes 2} \odot c < c \odot a \odot c \rightarrow c < a \odot c^{\otimes 2} \odot b^{\otimes 2}$.

So we can give the definition:

Definition 2.4.3. *The roots of a tropical polynomial $p(x) = a_0 \oplus a_1 \odot x \oplus \dots \oplus a_n \odot x^{\otimes n}$ are the tropical numbers x_0 for which exist a pair i, j such that $p(x_0) = a_i \odot x_0^{\otimes i} = a_j \odot x_0^{\otimes j}$ that is two monomials became equal or less than all the others.*

Example.

For example the second-degree tropical polynomial $3 \odot x^{\wedge 2} \oplus 2 \odot x \oplus 4$ can be factorize as $2 \odot (x \oplus (-1)) \odot (x \oplus 2)$ thus, its roots are the tropical numbers $x_1 = -1$ e $x_2 = 2$.

By substituting we get:

$$\begin{aligned} p(x_1) &= \min\{3 + 2x_1, 2 + x_1, 4\} = \min\{1, 1, 4\} = \min\{1, 4\} = \min\{2 + x_1, 1\} = 2 \odot x_1 \\ p(x_2) &= \min\{3 + 2x_2, 2 + x_2, 4\} = \min\{7, 4, 4\} = \min\{7, 4\} = \min\{2 + x_2, 1\} = 2 \odot x_2 \end{aligned}$$

Actually two monomials became less than or equal to the third.

OBSERVATION: we can notice that the roots of the polynomial $2 \odot (x \oplus (-1)) \odot (x \oplus 2)$ are -1 and 2 and not the opposite numbers as it happens in classical algebra.

Now we can also define the order of a root, but first remember its definition in classical mathematics:

Definition 2.4.4. *Given a polynomial $p(x)$ with coefficients in A and a root c of the polynomial, the order of c is the number k if $p(x)$ is a multiple of the polynomial $(x - c)^k$ but not of the polynomial $(x - c)^{k+1}$.*

In tropical geometry the idea comes from the fact that in order to have n different nodes all the straight lines $y = q_i + ix$ with $i = 0, \dots, n$ are needed and so two consecutive segments will always have slopes differing by 1. While, if a straight line does not contribute to the graph it will be skipped, and in the node the difference between the slopes of the consecutive segments will be greater than 1.

Definition 2.4.5. *The order of the tropical roots is given by the difference between the slopes of the two straight-line segments adjacent to the node.*

Example.

$$x^{\otimes 2} \oplus 3 \odot x \oplus 6 = (x \oplus 3) \odot (x \oplus 3) \quad \rightarrow \quad 3 \text{ is a root of order 2, that is a double root.}$$

Chapter 3

Tropical algebraic curves in \mathbb{R}^2

In classical mathematics $F(x; y) = 0$ is an algebraic equation which represents an algebraic curve if F represents a finite number of operations with the variables x and y , like addition, subtraction, multiplication, division and n root calculation.

Functions associated to polynomials of different degrees will represent different algebraic curves on the Cartesian plane.

Example:

$F(x; y)$ is a polynomial of first degree in x and y : the algebraic equation

$$F(x; y) = 0 \longrightarrow ax + by + c = 0$$

gives a straight line, unless $a = 0 \wedge b = 0$.

The points of an algebraic curve are the pairs (x, y) which satisfy the equation $F(x; y) = 0$ are exactly the points (x, y) , such that, substituted to the polynomial $F(x; y)$, yield zero.

How can we extend this concept in tropical geometry?

In classical mathematics we equal the polynomial $F(x; y)$ to zero - the neutral element of classical sum - but the neutral element of the minimum does not exist unless we consider $+\infty$.

If we operate in this way, we would consider tropical algebraic curves the set of points (x, y) such that, replaced to the polynomial $F(x; y)$, they make it $+\infty$. For example:

$$a \odot x^2 \oplus b \odot x \oplus c = +\infty$$

if and only if

$$a = +\infty \wedge b = +\infty \wedge c = +\infty \quad \text{or if} \quad x = +\infty \wedge c = +\infty$$

does not seem very interesting.

Furthermore, there is a difference in writing. The equation, in the function form $y = f(x)$ in classical mathematics, can be transformed in equation form $f(x) - y = 0$ and vice versa, while in tropical geometry this is not possible because only the second principle of equivalence can be applied.

The most interesting analogy between classical and tropical mathematics is the definition of the tropical algebraic curve as the set of tropical zeros of the polynomial $F(x; y)$.

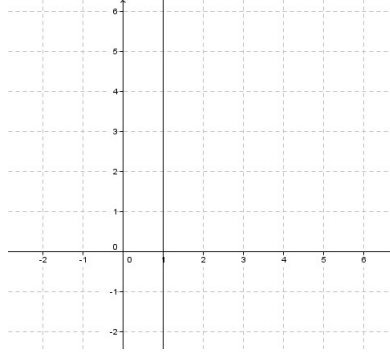
Definition 3.0.1. *Given a tropical polynomial $F(x; y)$ in two variables with coefficients in $\mathbb{R} \cup \{+\infty\}$, we define the tropical algebraic curve as the set of all the points $(x, y) \in \mathbb{R}^2$ which are the tropical zeros of $F(x; y)$.*

It means that in tropical geometry the algebraic curve is made up by the solutions of N equations obtained by equalling two monomials of a polynomial $F(x; y)$. N is equal to the k -combination $C_{n,k}$ with n the number of monomial of the polynomial and $k = 2$

Example.

Basing on the above definition, the tropical curve associated to the polynomial in two variables $3 \odot x \oplus 4$ (which coincides with $3 \odot x \oplus +\infty \odot y \oplus 4$) consists in the points $(x, y) \in \mathbb{R}^2$ such that they make the addends $(3 \odot x)$ and 4 equal. So every point $(1, y)$ for all $y \in \mathbb{R}$.

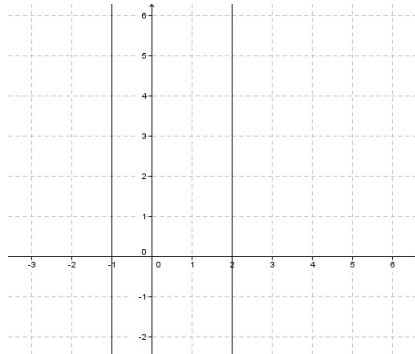
Here is the graph of the tropical polynomial of first degree $3 \odot x \oplus 4$:



Example.

The tropical algebraic curve associated to a polynomial in two variables $3 \odot x^2 \oplus 2 \odot x \oplus 4$ consists in the points $(x, y) \in \mathbb{R}^2$ such that they make two monomials equal and less than or equal to the third one.

We factorize the polynomial like this: $2 \odot (x \oplus (-1)) \odot (x \oplus 2)$ so its zeros are the tropical numbers $x_1 = -1$ e $x_2 = 2$. Therefore the algebraic curve consists in the points $(-1, y)$ or $(2, y)$ for all $y \in \mathbb{R}$.



The graph of a tropical polynomial of n degree will be made by n different vertical straight lines at the most.

It's difficult to name these curves as we do in classical geometry (straight lines, circumferences, parabolas, etc.). In fact the sketch depends on the number of monomials and not on the degree of the polynomial. In our mathematics we make a difference between the curves basing on the degree of the polynomial because that is important in the research of the roots. In tropical geometry we can distinguish the curves basing on the number of monomials of the polynomial since it is more relevant to find the tropical zeros than its degree.

3.1 Algebraic Curves associated with a monomial

To represent a polynomial we have to equal two of its monomials and make them less than or equal to all the others. In this case the polynomial is made by one monomial only, so we cannot form an equality and we have to add a second monomial that does not change the polynomial. Supposing to represent the polynomial $a \odot x$, the second term added must have $+\infty$ as coefficient because the infinity is the neutral element of the sum.

Therefore the polynomial becomes $a \odot x \oplus +\infty \odot y$.

To represent it, we have to make its two monomials equal, obtaining:

$$\begin{aligned} a + x &= +\infty + y \\ a + x &= +\infty \quad \forall y \\ x &= +\infty \quad \forall y \end{aligned}$$

So the monomial $a \odot x$ will be represented by a straight line parallel to the y -axis at an infinite distance.

We can show that $a \odot x \oplus +\infty \odot y$ is equal to $a \odot x \oplus +\infty \odot y \oplus +\infty$ and they are represented by the same graphic.

1.

$$\begin{aligned} a + x &= +\infty + y \\ x &= +\infty \quad \forall y \end{aligned}$$

Condition:

$$\begin{aligned} a + x &\leq +\infty \\ x &\leq +\infty \end{aligned}$$

Since $x = +\infty \wedge x \leq +\infty \implies x = +\infty$

2.

$$\begin{aligned} a + x &= +\infty \\ x &= +\infty \end{aligned}$$

Condition:

$$\begin{aligned} a + x &\leq y + \infty \\ x &\leq +\infty \end{aligned}$$

So also in this case $x = +\infty$

3.

$$\begin{aligned} +\infty + y &= +\infty \\ +\infty &= +\infty \implies \forall y \end{aligned}$$

Condition:

$$\begin{aligned} +\infty + y &\leq a + x \\ +\infty &\leq x \implies x = +\infty \end{aligned}$$

Therefore $a \odot x \oplus +\infty \odot y = a \odot x \oplus +\infty \odot y \oplus +\infty$. In fact both are represented by a straight line with infinite slope at infinite distance from y -axis.

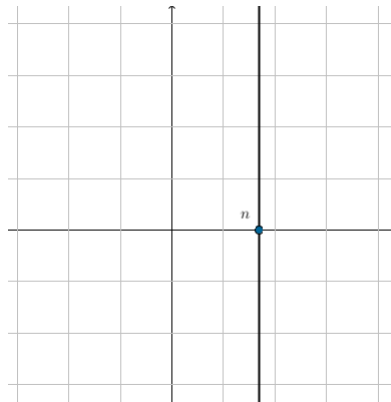
3.2 Algebraic Curves associated with a binomial

Now we'll consider some cases of curves made by two monomials. In this case, considering only polynomials made by two monomials, we simply have to make them equal, without comparing them with a third one.

Case $x \oplus n$

Let's consider the polynomial $x \oplus n$. In order to calculate its roots, we make its two monomials equal.
 $x = n, \forall y$.

So the graphic of this algebraic curve will be a straight line parallel to the y -axis.

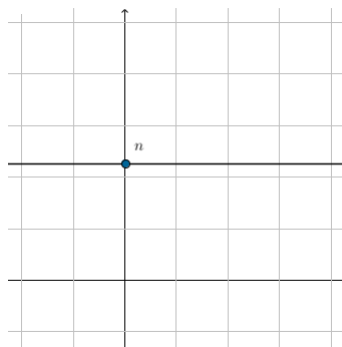


Case $y \oplus n$

Let's consider the polynomial $y \oplus n$. In order to calculate its roots we operate as we did in the previous case.

$$y = n, \forall x.$$

The graphic of this algebraic curve will be a straight line parallel to the x -axis.

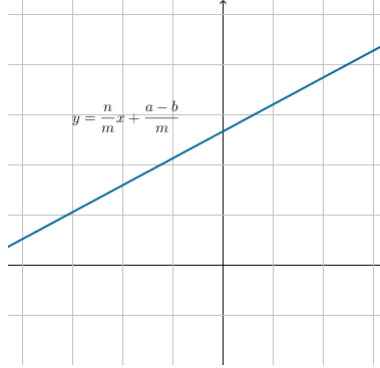


Case $a \odot x^{\otimes n} \oplus b \odot y^{\otimes m}$

Operating in the same way:

$$\begin{aligned}n \cdot x + a &= m \cdot y + b \\y &= \frac{n}{m}x + \frac{a-b}{m}\end{aligned}$$

So the roots of a polynomial, which is the sum of a monomial in x and one in y of any degree, are represented by a straight line with coefficient $\frac{n}{m}$ and intersection with y -axis in $\frac{a-b}{m}$.



In conclusion every polynomial made up by two monomials will always be a classical straight line.

3.3 Algebraic Curves associated with a trinomial

If we have polynomials made up by three monomials, we have to analyze three cases that come from the three possible equalities we can make.

Case $a \odot x^{\otimes n} \oplus b \odot y^{\otimes m} \oplus c$

This case is the generalization of the ellipses in classical mathematics and if $n = m = 1$ we have the tropical straight line.

1.

$$a + n \cdot x = b + m \cdot y$$

$$y = \frac{n}{m}x + \frac{a-b}{m}$$

Condition:

$$n \cdot x + a \leq c$$

$$x \leq \frac{c-a}{n}$$

This condition is equivalent to:

$$m \cdot y + b \leq c$$

$$y \leq \frac{c-b}{m}$$

2.

$$a + n \cdot x = c$$

$$x = \frac{c-a}{n}$$

Condition:

$$c \leq m \cdot y + b$$

$$y \geq \frac{c-b}{m}$$

3.

$$b + m \cdot y = c$$

$$y = \frac{c - b}{m}$$

Condition:

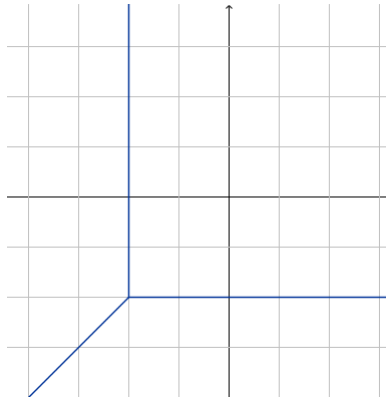
$$c \leq n \cdot x + a$$

$$x \geq \frac{c - a}{n}$$

So the representation of a polynomial that can be reduced to the form $a \oplus x^{\otimes n} \oplus b \oplus y^{\otimes m} \oplus c$ is represented by a ray parallel to x -axis, one parallel to y -axis and one with coefficient equal to the quotient between the coefficient of x and y . The three straight lines meet in the same point $P\left(\frac{c - a}{n}; \frac{c - b}{m}\right)$.

There are some particular cases of this polynomial. For example, when $n = m = 2$, we obtain our circumference. In this case and when $n = m$, the coefficient of the third straight line is 1, so it's the bisector of the first and the third quadrant.

For example the representation of the polynomial: $x^2 \oplus y^2 \oplus 4$ is:

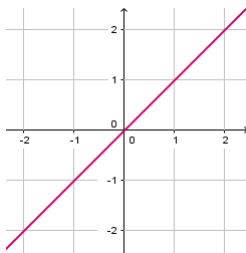


So it does not deal with the circumference of classical mathematics because it doesn't satisfy the definition of geometric locus (all the points belonging to a plane that is equidistant from a point).

Other examples:

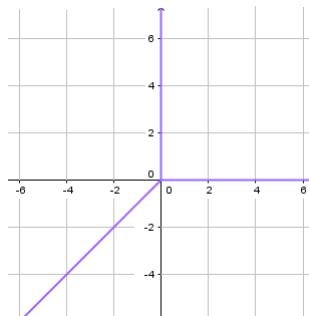
$$p(x, y) = x \oplus y$$

"tropical line"



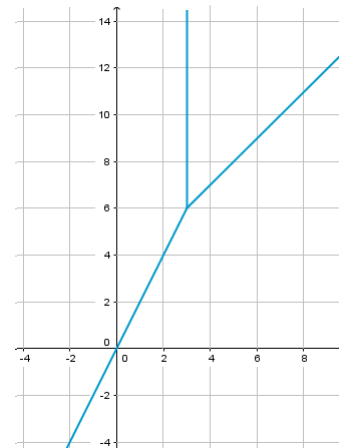
$$p(x, y) = x \oplus y \oplus 0$$

"tropical line"



$$p(x, y) = x^2 \oplus 3 \otimes x \oplus y$$

"tropical parabola"



Conclusions

In this work we built the structure $(\mathbb{R} \cup \{+\infty\}, \oplus, \odot)$ similarly to how standard number sets are constructed. We obtained a tropical semiring.

As \mathbb{C} , such semiring is algebraically closed, which means that the fundamental theorem of algebra holds in tropical geometry.

We studied how tropical polynomials can generate tropical algebraic curves (in \mathbb{R}^2 in particular) and we got that the curve essentially depends on the number of monomials that form a polynomial rather than the degree of such polynomial.

We drew some curves (curves of second degree in particular) and we saw that they do not look like the curves of classical mathematics, in fact they are formed of pieces of linear functions.

Bibliography

- [1] Appunti ed esercizi di Matematica Discreta, Roggero Margherita, Quaderni didattici del dipartimento di Matematica, University of Torino.
- [2] Speyer, D. and Sturmfels, B., "Tropical Mathematics", ArXiv Mathematics e-prints, math/0408099, Combinatorics, Algebraic Geometry, 2004, aug, <http://adsabs.harvard.edu/abs/2004math.....8099S>, Provided by the SAO/NASA Astrophysics Data System.
- [3] Brugallé, E. and Itenberg, I. and Mikhalkin, G. and Shaw, K., "Brief introduction to tropical geometry", ArXiv e-prints, "arXiv", 1502.05950, "math.AG", Mathematics - Algebraic Geometry, Mathematics - Combinatorics, 2015, feb, <http://adsabs.harvard.edu/abs/2015arXiv150205950B>, Provided by the SAO/NASA Astrophysics Data System.